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Theory 2

Exam 2

1) Given $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \theta)$, where μ is known and θ unknown. We are also given $T(\underset{\sim}{x}) = \sum_{i=1}^{n}(x_i - \mu)^2$ is a sufficient statistic for θ. Also, the probability density function for $T(\underset{\sim}{x})$ is $g(T(\underset{\sim}{x})|\theta) = \frac{1}{\Gamma(n/2)(2\theta)^{n/2}} t^{n/2 - 1} \exp\{-t/2\theta\}$, or in other words, $T \sim \text{gamma}(\frac{n}{2}, 2\theta)$. Lastly the MLE for θ is $\hat{\theta} = \frac{T}{n}$.

a) For $\theta_0 \in \mathbb{R} > 0$, we consider the hypotheses

$$H_0: \theta \leq \theta_0 \quad \text{vs.} \quad H_1: \theta > \theta_0$$

i) To derive the LRT, we first find the LRT statistic. Since $\{\Theta_0 : 0 < \theta \leq \theta_0\}$ is a set, the restricted MLE to derive $\lambda(\underset{\sim}{x})$ will depend on if θ_0 is above or below the true MLE. So then let

$$\lambda(\underset{\sim}{x}) = \begin{cases} \frac{L(\theta_0 | \underset{\sim}{x})}{L(\hat{\theta} | \underset{\sim}{x})} & \text{if } \theta_0 < \hat{\theta} \\ 1 & \text{if } \theta_0 \geq \hat{\theta} \end{cases} = \begin{cases} \frac{\exp\left\{\frac{-1}{2\theta_0}\sum_{i=1}^{n}(x_i - \mu)^2\right\}}{\exp\left\{-\frac{1}{2\hat{\theta}}\sum_{i=1}^{n}(x_i - \mu)^2\right\}} & \text{if } \theta_0 < \hat{\theta} \\ 1 & \text{if } \theta_0 \geq \hat{\theta} \end{cases}$$

Since $\hat{\theta} = \frac{\sum_{i=1}^{n}(x_i - \mu)^2}{n}$

$$= \begin{cases} \frac{\exp\left\{\frac{-1}{2\theta_0}\sum_{i=1}^{n}(x_i - \mu)^2\right\}}{\exp\left\{\frac{-n}{2}\right\}} & \text{if } \theta_0 < \hat{\theta} \\ 1 & \text{if } \theta_0 \geq \hat{\theta} \end{cases} = \begin{cases} \exp\left\{\frac{1}{2}\left(n - \frac{T(\underset{\sim}{x})}{\theta_0}\right)\right\} & \text{if } \theta_0 < \hat{\theta} \\ 1 & \text{if } \theta_0 \geq \hat{\theta} \end{cases}$$

Since $T(\underset{\sim}{x}) = \sum_{i=1}^{n}(x_i - \mu)^2$

So the LRT is the test that rejects H_0 for $\{\underset{\sim}{x} : \lambda(\underset{\sim}{x}) < c\}$ such that $c \in [0, 1]$.

ii)

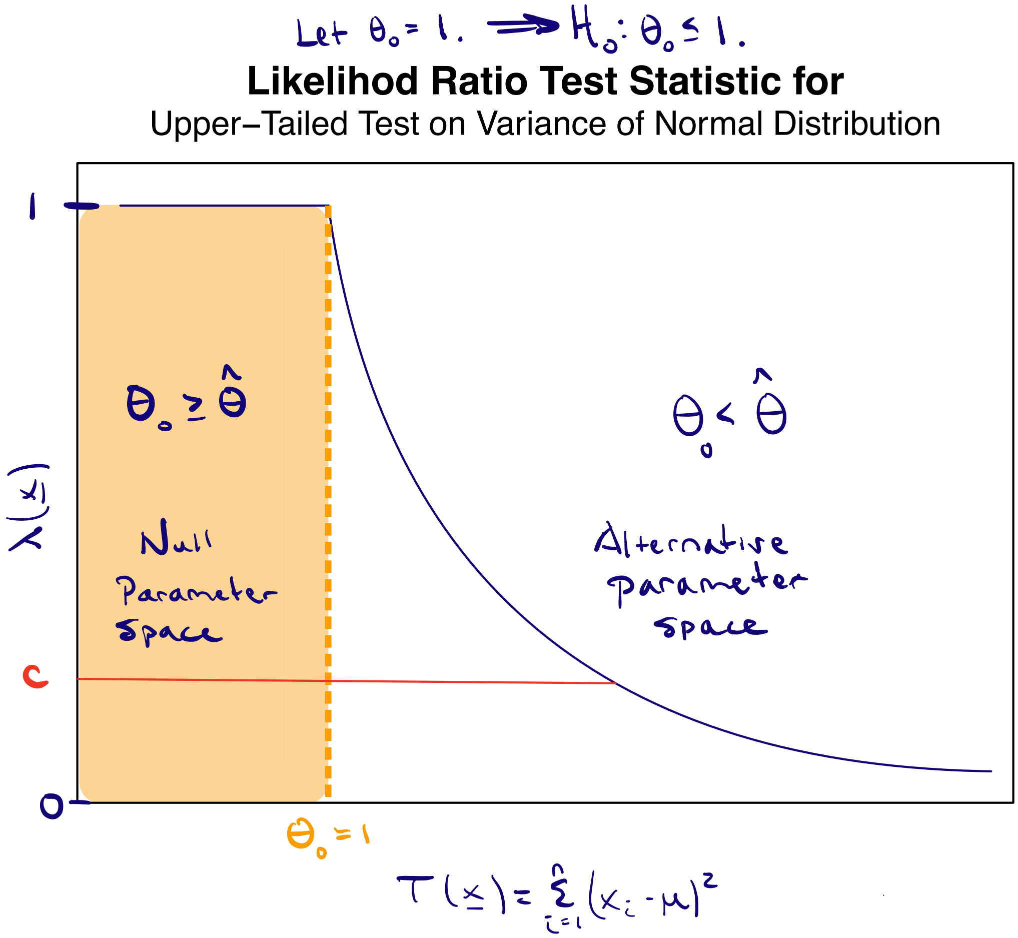

Figure 1: Axes for graph of likelihood ratio test statistic.

iii) So to find the UMP test, we can apply the Karlin-Rubin theorem if we show that $g(T(\underset{\sim}{x})|\theta)$ has monotone likelihood ratio (MLR). So then let θ_1 and θ_2 be elements of the parameter space Θ such that $\theta_2 > \theta_1$.

Then we have that

$$\frac{g(t|\theta_2)}{g(t|\theta_1)} = \frac{\frac{1}{\Gamma(n/2)(2\theta_2)^{n/2}} t^{n/2-1} \exp\{-t/2\theta_2\}}{\frac{1}{\Gamma(n/2)(2\theta_1)^{n/2}} t^{n/2-1} \exp\{-t/2\theta_1\}}$$

$$= \left(\frac{2\theta_1}{2\theta_2}\right)^{n/2} \exp\left\{-t\left(\frac{1}{2\theta_2} - \frac{1}{2\theta_1}\right)\right\}.$$

which is monotone over the parameter space, therefore the gamma$(\frac{n}{2}, 2\theta)$ family has MLR.

So by the Karlin-Rubin Theorem given sufficient statistic $T(\underset{\sim}{x}) = \sum_{i=1}^{n}(x_i - \mu)^2$ whose distribution has monotone likelihood ratio, the UMA α-level test is the test that rejects H_0 if $T(\underset{\sim}{x}) > K\theta_0$, where K satisfies $P_{\theta_0}(T > K\theta_0) = \alpha$. We know that $T \sim \text{gamma}(\frac{n}{2}, 2\theta_0) \Rightarrow \frac{T}{\theta_0} \sim \chi^2_n$, since we presume H_0. So then K satisfies $P_{\theta_0}(\frac{T}{\theta_0} > K) = \alpha$, and is the $\underline{\alpha}$ quantile of the χ^2_n distribution. So then
upper tail notation
the rejection region for this α-level UMP test is

$$\{\underset{\sim}{x} : T(\underset{\sim}{x}) > \chi^2_{n,\alpha}\theta_0\}.$$

iv) The power function is the probability of rejecting the null hypothesis over the parameter space.

(i.e. $\beta(\theta) = P_\theta\left(\underline{X} \in \{\underline{x} : T(x) > \chi^2_{n,\alpha}\theta_0\}\right)$).

So $\beta(\theta) = P_\theta\left(T(\underline{X}) > \chi^2_{n,\alpha}\theta_0\right)$.

v.) Let $\theta_0 = 1$. $\Longrightarrow H_0: \theta_0 \le 1$.

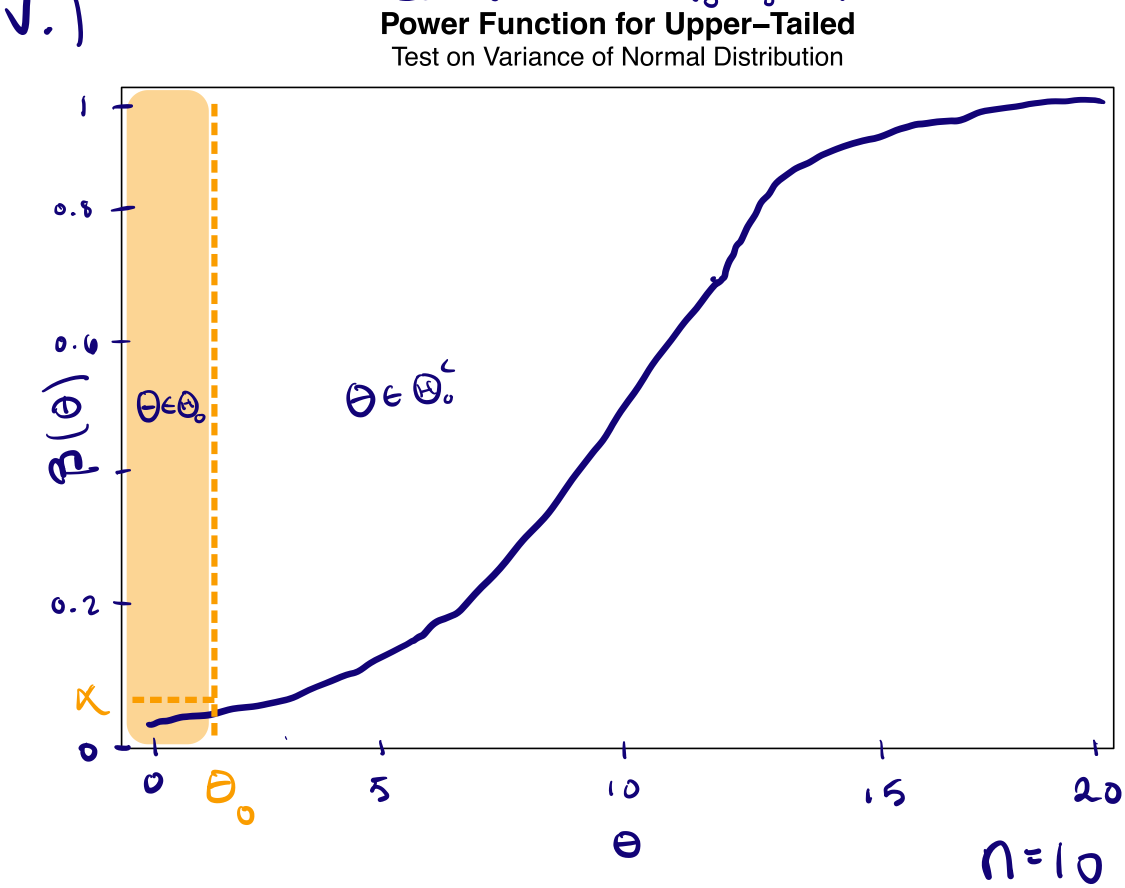

Figure 2: Axes for graph of power function.

vi) To find a valid p-value, we must find a p-value such that for every $\theta \in \Theta_0$ and every $\alpha \in [0,1]$, then $P(p(\underline{x}) \le \alpha) \le \alpha$. Let the p-value be defined as $p(\underline{x}) = P(T(\underline{x}) > \chi^2_n \theta_0) = P\left(\frac{T(\underline{x})}{\theta_0} > \chi^2_{n,\alpha}\right)$.
By definition, this p-value is a valid p-value because the χ^2_n α quantile satisfies $p(\underline{x}) = \alpha$. so $P(p(\underline{x}) \le \alpha) \le \alpha$. So $p(\underline{x}) = P\left(\frac{T(\underline{x})}{\theta_0} > \chi^2_{n,\alpha}\right)$ is a valid p-value.

b) To find the UMA most accurate lower confidence bound for θ, we can invert the acceptance region for the α-level UMP test. From the α-level UMP test in 1(a)(iii), we have that the acceptance region is

$$A(\underline{x}) = \{\underline{x} : T(\underline{x}) \le \chi^2_{n,\alpha}\theta_0\}.$$

Inverting this acceptance region yields the $1-\alpha$ confidence set $C(\theta) = \left\{\theta : \theta \ge \frac{T(\underline{x})}{\chi^2_{n,\alpha}}\right\}$.

So the $1-\alpha$ UMA confidence bound for θ is $\frac{T(\underline{x})}{\chi^2_{n,\alpha}}$.

2) Given $X_1, \dots, X_n \overset{iid}{\sim} N(\theta, \sigma^2)$, σ^2 known. Also, let $\theta \sim N(\mu, \tau^2)$, which implies $\theta | \underline{x} \sim N\left(\frac{n\bar{x}\tau^2 + \mu\sigma^2}{n\tau^2 + \sigma^2}, \frac{\sigma^2\tau^2}{n\tau^2 + \sigma^2}\right)$.

a) Considering the hypotheses $H_0: \theta \leq \theta_0$ vs. $H_1: \theta > \theta_0$, The bayesian test is the test that would reject H_0 if

$$P(\theta < \theta_0 | \underline{x}) < \frac{1}{2}.$$

So then if

$$\int_{-\infty}^{\theta_0} \pi(\theta | \underline{x}) \, d\theta < \frac{1}{2},$$

then H_0 is rejected.

b) To find the $1-\alpha$ Bayesian HPD for θ, we define two values as the endpoints of our credible set (θ_L, θ_U) that satisfy

$$1 - \alpha = \int_{\theta_L}^{\theta_U} \pi(\theta | \underline{x}) \, d\theta.$$

Since our posterior is normal, it is unimodal, which implies if $\pi(\theta_L | \underline{x}) = \pi(\theta_U | \underline{x})$, then the set (θ_L, θ_U) is the HPD credible set for θ.

Normal is location-scale family

Consider the transformation: since $\theta | \underline{x} \sim N\left(\frac{n\bar{x}\tau^2 + \mu\sigma^2}{n\tau^2+\sigma^2}, \frac{\sigma^2\tau^2}{n\tau^2+\sigma^2}\right)$, we can then write:

$$\frac{\theta - \left(\frac{n\bar{x}\tau^2+\mu\sigma^2}{n\tau^2+\sigma^2}\right)}{\left(\frac{\sigma^2\tau^2}{n\tau^2+\sigma^2}\right)} = \frac{\theta n\tau^2 + \theta\sigma^2 - n\bar{x}\tau^2 + \mu\sigma^2}{\sigma^2\tau^2} = \frac{n\tau^2(\theta - \bar{x}) + \sigma^2(\theta+\mu)}{\sigma^2\tau^2} \sim N(0,1).$$

which implies θ_L, θ_U would need to be standard normal quantiles, $z_{\frac{1-\alpha}{2}}, z_{\alpha/2}$ respectively (upper tail notation).

So then we can write the $1-\alpha$ credible set as

$$\left\{\theta : \frac{n\tau^2(z_{\frac{1-\alpha}{2}} - \bar{x}) + \sigma^2(\theta+\mu)}{\sigma^2\tau^2} < \theta < \frac{n\tau^2(z_{\frac{\alpha}{2}} - \bar{x}) + \sigma^2(\theta+\mu)}{\sigma^2\tau^2}\right\}.$$

c) If a posterior distribution is symmetric about its mean, then a HPD and equal tail credible set are the same set for a given problem. If a posterior distribution is skewed, however, then the HPD finds the interval with the HPD where $\pi(\theta_L|\underline{x}) = \pi(\theta_U|\underline{x})$, which does not guarantee equal area beyond the tails, whereas the the equal tail credible set imposes a restriction that the area under $\pi(\theta|\underline{x})$ beyond either ends of the interval must be equal.